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Examples of underdetermined scattering problems

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Abstract. The determination of the shape of a scatterer by non-destructive methods, such as scattering experiments, raises the question of indetermination or ambiguity problems for the scatterer's shape, depending on the given scattering data. These ambiguity problems are discussed here by means of numerical constructions of equivalent scattering problems, i.e. we determined scatterers that produced the same scattering amplitude in given conditions (fixed energies, fixed incident angles and/or directions of receivers). This construction is given in the context of a generalized scattering theory that takes into account impedance discontinuities inside the scatterer. We started with a scattering problem defined by a discontinuous curve of arbitrary shape and defined by boundary conditions. Then, a circular curve with appropriate boundary conditions was determined such that these two scattering problems yielded the same scattering amplitude within the given conditions. We also present numerical results for two particular cases of the generalized scattering theory, calculated by means of the Nyström method. We used these results to verify that the equivalence obtained in the Born approximation holds for the exact scattering amplitudes.

1. Introduction

From previous results [1, 2], we know that the reconstruction of a finite scatterer by scalar wave diffusion is unique if the scattering data are known for all illumination angles and all receiver directions at a fixed energy. We have already shown [3] that if these conditions are weakened, we do not keep uniqueness of the shape determination. This paper gives further examples and demonstrations of this point. In particular, we present a different ambiguous case which contains the backscattering. We use the scattering theory in the Sabatier framework [4] corresponding to an impedance equation with discontinuity curves corresponding to a jump in the impedance and/or its normal derivative. We specialize to the two-dimensional case as in [3]. Between these curves, the scatterer is assumed to be inhomogeneous, only allowing a smooth variation of parameters, such that the impedance is twice differentiable. We consider this model concentrating on the part that takes only discontinuities into account. In section 4 of [3], we showed that this restriction was relevant. Here we present a numerical treament of two particular cases of an impedance equation without diffuse scattering and we calculate the exact scattering amplitude using the Nyström method, with a particular decomposition used by Kress [5] in order to isolate the logarithmic singularities of the two-dimensional case. Finally, we present a construction method for equivalent scattering problems and discuss numerical examples.

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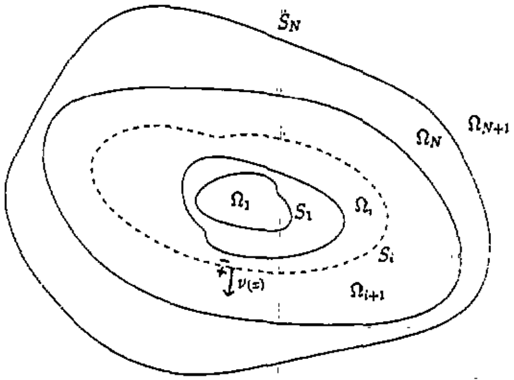

Figure 1. Scattering problem with discontinuities: shape of discontinuities.

2. The scattering problem

We start from the impedance equation

$$(\alpha^{-2}\,\mathrm{div}\,\alpha^2\,\mathrm{grad} + k^2 - V(x))\varphi(k,x) = 0 \tag{2.1}$$

where $k, x \in \mathbb{R}^2$, $\alpha > 0$ is defined inside domains $\Omega_1, \Omega_2, \ldots, \Omega_{N+1}$, such that $\Omega_i \bigcap \Omega_j = \emptyset$ for any $i \neq j$, $\mathbb{R}^2 = \sum_0^{N+1} \overline{\Omega}_i$. S_i is the external boundary of Ω_i and the internal boundary of Ω_{i+1}. The domains are ordered from Ω_1 to Ω_{N+1} and are all finite except Ω_{N+1}, which extends to infinity in all directions (see figure 1). In addition we assume that each S_i is C^2 and that α is C^2 inside $\mathbb{R}^2\backslash S$, where $S = \cup_{i=1}^N S_i$, with $\alpha(x)$ and $\partial\alpha(x)/\partial\nu_x$ going to finite limits at any point $x_i \in S_i$ as $x \longrightarrow x_i$ inside the domain Ω_i or Ω_{i+1}, where ν is a vector normal to S_i and pointing outwards, i.e. in Ω_{i+1}. At any point $x_p \in S_p$, by labelling the + and − sides of ν_p as external and internal parts, we can characterize the jump of α and its derivative throughout S_p by the following 'singular data':

(i) Transmission and reflection factors

$$\frac{1}{t_p} = \frac{1}{2}\left[\frac{\alpha_p^+}{\alpha_p^-} + \frac{\alpha_p^-}{\alpha_p^+}\right] \frac{r_p}{t_p} = \frac{1}{2}\left[\frac{\alpha_p^+}{\alpha_p^-} - \frac{\alpha_p^-}{\alpha_p^+}\right].$$

(ii) Slope factor

$$\frac{\overline{s}_p}{t_p} = \frac{1}{2}\nu\left[\frac{\mathrm{grad}\,\alpha_p^-}{\alpha_p^+} - \frac{\mathrm{grad}\,\alpha_p^+}{\alpha_p^-}\right].$$

The impedance scattering problem was studied by Sabatier [4] in the three-dimensional case. Here we present the results for the two-dimensional case [3], first we have the following theorems.

Theorem 1. φ is a solution of equation (2.1) if, and only if, the function $\psi := \alpha\varphi$ is a solution of the chain of Schrödinger equations

$$(\Delta + k^2 - V - \alpha^{-1}\Delta\alpha)\psi(k,x) = 0 \qquad x \in \mathbb{R}^2\backslash S \tag{2.2}$$

coupled with the condition of continuity of ψ/α and $\alpha(\partial\psi/\partial\nu) - \psi(\partial\alpha/\partial\nu)$ through S.

Because of this theorem, there always exist two equivalent formulations of the same physical problem and it is useful to go back and forth from one to the other. In the *impedance formulation of the scattering problem, $\varphi(k,x)$ is a solution of*

$$(\alpha^{-2}\operatorname{div}\alpha^2\operatorname{grad}+k^2-V(x))\varphi(k,x)=0 \qquad x\in\mathbb{R}^2 \tag{2.3}$$

$\varphi_s := \alpha(x)\varphi(k,x)-\exp[ik\cdot x]$ is Sommerfeld outgoing, i.e. in two-dimensional cases:

$$\left(\frac{x}{|x|}\cdot\operatorname{grad}\varphi_s(x)\right)-ik\varphi_s(x)=o\left(\frac{1}{|x|^{1/2}}\right) \qquad |x|\to\infty. \tag{2.4}$$

In the Schrödinger chain formulation, $\psi(k,x)$ is a solution of

$$(\Delta+k^2-V-\alpha^{-1}\Delta\alpha)\psi(k,x)=0 \qquad x\in\mathbb{R}^2\backslash S \tag{2.5}$$

$$\alpha^+(x)\frac{\partial\psi^+(x)}{\partial\nu}-\psi^+(x)\frac{\partial\alpha^+(x)}{\partial\nu}=\alpha^-(x)\frac{\partial\psi^-(x)}{\partial\nu}-\psi^-(x)\frac{\partial\alpha^-(x)}{\partial\nu}$$

$$\psi^+(x)/\alpha^+(x)=\psi^-(x)/\alpha^-(x) \qquad x\in S \tag{2.6}$$

$$\psi(k,x)-\exp[ik\cdot x] \text{ is Sommerfeld outgoing.} \tag{2.7}$$

We recall that this Schrödinger chain formulation was used to derive a generalized Lipmann–Schwinger equation and a scattering amplitude in two parts [4]. One part is due to the discontinuities (related to (2.6), and denoted A_0); the other is due to diffuse scattering in the presence of discontinuities (related to the potential $V+\alpha\Delta\alpha$, and denoted A_1).

In this paper we consider only the part due to the discontinuities. Using the same notation as in the previous paper, we denote ψ_{in} as the solution to the scattering problem *with only discontinuities, i.e. the solution of the following system:*

$$\begin{aligned}&(\Delta+k^2)\psi_{\text{in}}(k,x)=0 \qquad x\in\mathbb{R}^2\backslash S\\ &\psi_{\text{in}}/\alpha \text{ and } \alpha\frac{\partial\psi_{\text{in}}}{\partial\nu}-\psi_{\text{in}}\frac{\partial\alpha}{\partial\nu} \text{ continuous}/S\\ &\psi_{\text{in}}(x,y)-\exp[ik\cdot x] \text{ is Sommerfeld outgoing.}\end{aligned} \tag{2.8}$$

In order to solve this system we write $u(x):=\psi_{\text{in}}(k,x)-e^{ikx}$, with

$$u(x)=\sum_{j=1}^{N}\int_{S_j}ds(z)\,\Phi(z,x)\phi_j(z)+\sum_{j=1}^{N}\int_{S_j}ds(z)\frac{\partial\Phi}{\partial\nu_z}(z,x)\psi_j(z) \tag{2.9}$$

where $\Phi(x,y)$ is the Helmholtz–Green function, i.e.

$$\Phi(x,y)=\frac{i}{4}H_0^{(1)}(|k||x-y|) \qquad x,y\in\mathbb{R}^2,\ x\neq y. \tag{2.10}$$

As in the three-dimensional case, $u(x)$ satisfies (2.8) as well as the Sommerfeld condition. We determine ϕ_j and ψ_j by the continuity conditions (2.8), obtaining

$$\begin{cases}\psi=2\beta e^{ik\cdot x}+\beta\boldsymbol{S}\phi+\beta\boldsymbol{K}\psi\\ \phi+\gamma\psi=2\beta\dfrac{\partial e^{ik\cdot x}}{\partial\nu}-2\beta' e^{ik\cdot x}+\beta\boldsymbol{T}\psi-\beta'\boldsymbol{K}\psi+\beta\boldsymbol{K}'\phi-\beta'\boldsymbol{S}\phi\end{cases} \tag{2.11}$$

with

$$\psi_j = \psi \qquad \text{for } x \in S_j$$

$$\beta(x) = \frac{\alpha^+(x) - \alpha^-(x)}{\alpha^+(x) + \alpha^-(x)} \qquad \beta'(x) = \frac{\alpha'^+(x) - \alpha'^-(x)}{\alpha^+(x) + \alpha^-(x)} \qquad \gamma(x) = \frac{\alpha'^+(x) + \alpha'^-(x)}{\alpha^+(x) + \alpha^-(x)}$$

and the surface operators $\boldsymbol{S}$, $\boldsymbol{K}$, $\boldsymbol{K}'$, $\boldsymbol{T}$ are defined by

$$(S_{ij} f)(x) = 2\int_{S_j} \mathrm{d}s(z)\, \Phi(z,x) f(z) \qquad x \in S_i \tag{2.12}$$

$$(K_{ij} f)(x) = 2\int_{S_j} \mathrm{d}s(z)\, \frac{\partial \Phi}{\partial \nu_z}(z,x) f(z) \qquad x \in S_i \tag{2.13}$$

$$(K'_{ij} f)(x) = 2\int_{S_j} \mathrm{d}s(z)\, \frac{\partial \Phi}{\partial \nu_x}(z,x) f(z) \qquad x \in S_i \tag{2.14}$$

$$(T_{ij} f)(x) = 2\frac{\partial}{\partial \nu_x}\int_{S_j} \mathrm{d}s(z)\, \frac{\partial \Phi}{\partial \nu_z}(z,x) f(z) \qquad x \in S_i. \tag{2.15}$$

The following result is essential:

Theorem 2. We assume that β, β', γ, ψ are in $C^1(S)$ and ϕ in $C(S)$. If $N[1-\boldsymbol{B}/(1+\beta^2)] = 0$ and $N(1-\beta\boldsymbol{K}) = 0$, then the system (2.11) has a unique solution

$$\psi = (\mathbf{1} - \beta\boldsymbol{K})^{-1}\beta(2\mathrm{e}^{\mathrm{i}k\cdot x} + \boldsymbol{S}\phi) \tag{2.16}$$

$$\phi = [(1+\beta^2)\mathbf{1} - \boldsymbol{B}]^{-1}A\mathrm{e}^{\mathrm{i}k\cdot x} \tag{2.17}$$

with

$$A\mathrm{e}^{\mathrm{i}k\cdot x} = 2\beta\frac{\partial \mathrm{e}^{\mathrm{i}k\cdot x}}{\partial \nu} - 2[\beta' + (\gamma\mathbf{1} + \beta'\boldsymbol{K} - \beta\boldsymbol{T})(1-\beta\boldsymbol{K})^{-1}\beta]\mathrm{e}^{\mathrm{i}k\cdot x} \tag{2.18}$$

$$\boldsymbol{B} = -(\gamma\mathbf{1} + \beta'\boldsymbol{K})(1-\beta\boldsymbol{K})^{-1}\beta\boldsymbol{S} + \beta\boldsymbol{K}' - \beta'\boldsymbol{S} + \beta\boldsymbol{C} \tag{2.19}$$

$$\boldsymbol{C} = \boldsymbol{T}(\mathbf{1} - \beta\boldsymbol{K})^{-1}\beta\boldsymbol{S} + \beta\mathbf{1}. \tag{2.20}$$

Proof. To show this result, we used the Riesz–Fredholm theory for compact operators. The uniqueness is assured because the kernels $N[1-(\boldsymbol{B}/1+\beta^2)]$ and $N(1-\beta\boldsymbol{K})$ are null. The existence for ψ is guaranteed since $\beta\boldsymbol{K}$ is a compact operator. For ϕ, the problem is with the operator $\boldsymbol{T}$, as it is not bounded, but here it is shown as a product with other operators. Sabatier has demonstrated that $(1+\beta^2)^{-1}\boldsymbol{B}$ is a compact operator. For more details see [4]. □

Now we define the scattering amplitude A_0, which is related to ψ_{in} and is due to the discontinuities only. To obtain it, we apply Green's theorem to $\psi_{\mathrm{in}}(k,y)$ and $\Phi(x,y)$ inside the domain defined by $|y| \leqslant R$ and $y \in \Omega_{N+1}$, before we let $R \to \infty$, and we obtain

$$\psi_{\mathrm{in}}(k,x) = \mathrm{e}^{\mathrm{i}k\cdot x} - \frac{1}{(8\pi|k|)^{1/2}}\frac{\mathrm{e}^{\mathrm{i}[|k||x|+(\pi/4)]}}{|x|^{1/2}}A_0(|k|\hat{x},k) + o\left(\frac{1}{|x|^{1/2}}\right) \tag{2.21}$$

with

$$A_0(k', k) = \int_{S_N^+} e^{-ik'\cdot y}\left[\frac{\partial \psi_{in}(k, y)}{\partial \nu_y} + ik'\cdot \nu(y)\psi_{in}(k, y)\right] ds(y) + o\left(\frac{1}{|x|^{1/2}}\right) \tag{2.22}$$

where $k' := k\hat{x}$.

There is another expression for A_0 in terms of ϕ and ψ. To obtain it, one puts the asymptotic expansions of $\Phi(x, y)$ and $\partial\Phi(x, y)/\partial\nu_x$ inside the definition (2.9) of u, obtaining

$$\begin{aligned}\phi_{in}(k, x) = e^{ik\cdot x} &+ \sum_{j=1}^{N} \frac{1}{(8\pi k)^{1/2}} \frac{e^{i[k|x|+(\pi/4)]}}{|x|^{1/2}} \int_{S_j} ds(y)\, e^{-ik\hat{x}\cdot y}\phi(y)\\ &- \sum_{j=1}^{N} \frac{i}{(8\pi k)^{1/2}} \frac{e^{i[k|x|+(\pi/4)]}}{|x|^{1/2}} \int_{S_j} ds(y)\,(\nu_y \cdot k\hat{x})e^{-ik\hat{x}\cdot y}\psi(y).\end{aligned} \tag{2.23}$$

Together with (2.21), one finally gets

$$A_0(k', k) = i\sum_{j=1}^{N} \int_{S_j} ds(y)\,(\nu_y \cdot k')e^{-ik'\cdot y}\psi(y) - \sum_{j=1}^{N} \int_{S_j} ds(y)\, e^{-ik'\cdot y}\phi(y) \tag{2.24}$$

where $k' = k\hat{x}$.

It is possible to obtain the calculations of the first- and second-order terms with respect to the potential's size and to r_p, s_p, t_p. In fact, with the assumptions of theorem 2, we can derive the following expansion up to second order in ψ and ϕ.

$$\begin{aligned}\phi &= 2\beta\frac{\partial e^{ik\cdot x}}{\partial \nu} - 2\beta' e^{ik\cdot x} + 2(\beta K'\beta - \beta' S\beta)\frac{\partial e^{ik\cdot x}}{\partial \nu}\\ &\quad + 2(\beta T\beta - \beta K'\beta' - \beta' K\beta + \beta' S\beta')e^{ik\cdot x} + O(\|\beta\|^3)\\ \psi &= 2\beta e^{ik\cdot x} + 2(\beta K\beta - \beta S\beta')e^{ik\cdot x} + 2\beta S\beta\frac{\partial e^{ik\cdot x}}{\partial \nu} + O(\|\beta\|^3).\end{aligned} \tag{2.25}$$

We get the scattering amplitude at the first (Born approximation) and second order (quadratic approximation), by using relation (2.24) with these expansions of ψ and ϕ.

The result can then be reduced to the simplest form *by using the standard equivalence* [4], which in the Born approximation is

$$A_0^B(k', k) = -2\sum_{j=0}^{N} \int_{S_j} ds(z)\,\beta(z)\frac{\partial}{\partial \nu_z}[e^{i(k-k')\cdot z}] + 2\sum_{j=0}^{N} \int_{S_j} ds(z)\,\beta'(z)e^{i(k-k')\cdot z}. \tag{2.26}$$

3. Numerical resolution of two particular cases

Now we present a numerical solution for ψ_{in} in two special cases. In the first, only $\partial\alpha/\partial\nu$ is discontinuous and in the second the relative discontinuity $\alpha^+(x)/\alpha^-(x)$ does not depend on the position of x on the surface. We recall that the general case, where ϕ and ψ are defined by (2.17) and (2.16), is much more difficult to solve because the operator $\boldsymbol{T}$ (in particular T_{ii}) is not compact; a problem which must be circumvented.

In order to solve the integral equations, we use the Nyström method and a particular decomposition of their kernels to circumvent the logarithmic singularity. This is done by applying the Kress method [5, 6] (a numerical treatment of exterior boundary-value problems for the Helmholtz equation expressed by means of integral equations).

First, we give results for the unicity and existence of ψ_{in} for these particular cases. We then present the decomposition of kernels to be used in the Nyström method. Finally, we give the systems that are to be solved numerically and a convergence result.

3.1. The two particular cases

The first case is $\alpha \in C(\mathbb{R}^2)$. In this case we have $\beta = 0$ and from equations (2.17) and (2.16) we obtain the following system

$$\psi = 0 \qquad (1 + \beta' S)\phi = -2\beta' \mathrm{e}^{\mathrm{i}k\cdot x}. \tag{3.27}$$

Theorem 3. If $\beta' \in C(S)$ and S is a bounded surface belonging to the class C^2 then $(1 + \beta' S)^{-1}$ exists as a bounded operator on $C(S)$. The solution of the system (3.27) is

$$\psi = 0 \qquad \phi = -2(1 + \beta' S)^{-1}\beta' \mathrm{e}^{\mathrm{i}k\cdot x}. \tag{3.28}$$

Proof. We also applied the Riesz–Fredholm theory for compact operators. Using the result for the homogenous chain in [4], we know that if there exists a solution, it is unique. The existence holds in general because $\beta' S$ is a compact operator on $C(S)$ [2]. □

For the second case, the relative discontinuity $\alpha^+(x)/\alpha^-(x)$ does not depend on the position of x on the surface. We introduce the function $\sigma(x)$ defined by

$$\sigma_{N+1}(x) = 1 \qquad x \in \Omega_{N+1}$$

$$\sigma_i(x) = \sigma_{i+1}(x)\alpha^-(x)/\alpha^+(x) \qquad x \in \Omega_i \qquad i = 1, \ldots, N.$$

So to find ψ_{in}, we pose $v := u/\sigma$ which solves the following system:

$$(\Delta_x + k^2)v(x) = 0 \qquad x \in \mathbb{R}^2 \backslash S \tag{3.29}$$

$$v(x) \text{ and } \sigma^2 \left[\frac{v}{\sigma}\frac{\partial \alpha}{\partial \nu} - \frac{\partial v}{\partial \nu}\right](x) \text{ continuous through } S \tag{3.30}$$

$$v(x) \text{ is Sommerfeld outgoing (2.4).} \tag{3.31}$$

We seek $v(x)$ by using a single- and double-layer potential ansatz, which can be written as

$$v(x) = \sum_{j=1}^{N} \int_{S_j} \mathrm{d}s(z)\, \Phi(z, x)\phi_j(z) + \sum_{j=1}^{N} \int_{S_j} \mathrm{d}s(z) \frac{\partial \Phi}{\partial \nu_z}(z, x)\psi_j(z) \tag{3.32}$$

and which satisfies (3.29) and (2.4). We obtain the following system by then imposing the continuity conditions (3.30).

$$\psi = 0 \tag{3.33}$$

$$(1 - \tilde{\beta} K' + \tilde{\beta}' S)\phi = -2\tilde{\beta}' \mathrm{e}^{\mathrm{i}k\cdot x} + 2\tilde{\beta} \frac{\partial \mathrm{e}^{\mathrm{i}k\cdot x}}{\partial \nu_x} \tag{3.34}$$

with

$$\tilde{\beta} = \frac{\sigma_+^2 - \sigma_-^2}{\sigma_+^2 + \sigma_-^2} \qquad \text{and} \qquad \tilde{\beta}' = \frac{(\sigma_+^2/\alpha_+)(\partial\alpha_+/\partial\nu) - (\sigma_-^2/\alpha_-)(\partial\alpha_-/\partial\nu)}{\sigma_+^2 + \sigma_-^2}.$$

Notice that $\tilde{\beta}$ does not depend on the position on the curve but $\tilde{\beta}'$ does in general. As in the previous case, we have a result of existence and uniqueness.

Theorem 4. If $\tilde{\beta}, \tilde{\beta}' \in C(S)$ and S is a bounded surface belonging to the class C^2 then $(1 - \tilde{\beta}\boldsymbol{K}' + \tilde{\beta}'\boldsymbol{S})^{-1}$ exists and is a bounded operator on $C(S)$. The solution of the system (3.34) is

$$\psi = 0$$
$$\phi = -2(1 - \tilde{\beta}\boldsymbol{K}' + \tilde{\beta}'\boldsymbol{S})^{-1} \left(\tilde{\beta}' \mathrm{e}^{\mathrm{i}k\cdot x} + 2\tilde{\beta}\frac{\partial \mathrm{e}^{\mathrm{i}k\cdot x}}{\partial \nu_x}\right). \tag{3.35}$$

Proof. The proof is similar to the previous one. Recall that $\tilde{\beta}\boldsymbol{K}'$ and $\tilde{\beta}'\boldsymbol{S}$ are compact operators. □

Remark. We recall that the previous results are valid for more than one curve. The following numerical method is also correct for more than one curve, but, for the sake of simplicity, we take only one. If we take several curves, the principal difficulty is only the size of the linear system to resolve, which increases quickly with the number of the curve. In effect, if you have a square matrix $m \times m$ for one curve, you obtain a matrix $pm \times pm$ for p curves.

3.2. Numerical resolution

For the numerical resolution, we assume that the curve S is star-shaped and belongs to the class C^2. Its equation is of the form $R = f(\theta)$, where f is a 2π periodic function. Then for two points x, y of S we can write

$$x := \begin{pmatrix} f(\theta_x)\cos\theta_x \\ f(\theta_x)\sin\theta_x \end{pmatrix} \qquad \text{and} \qquad y := \begin{pmatrix} f(\theta_y)\cos\theta_y \\ f(\theta_y)\sin\theta_y \end{pmatrix}.$$

Now, to solve the integral equations (3.27) and (3.34) we isolate the logarithmic singularities. By using Kress notation [5], we obtain the following parametric form of $\boldsymbol{S}$ and $\boldsymbol{K}'$:

$$\boldsymbol{S}\phi(x) = \int_0^{2\pi} M(x, y)\phi(y)\,\mathrm{d}\theta_y \tag{3.36}$$

$$-\tilde{\beta}\boldsymbol{K}'\phi(x) = \tilde{\beta}\int_0^{2\pi} L'(x, y)\phi\,\mathrm{d}\theta_y \tag{3.37}$$

with

$$M(x, y) = \frac{\mathrm{i}}{2} H_0^{(1)}(kr(x, y))\sqrt{f^2(\theta_y) + f'^2(\theta_y)} \tag{3.38}$$

$$L'(x, y) = \frac{\mathrm{i}k}{2}[f^2(\theta_x) - f(\theta_x)f(\theta_y)\cos(\theta_y - \theta_x) + f'(\theta_x)f(\theta_y)\sin(\theta_y - \theta_x)]$$
$$\times \sqrt{\frac{f^2(\theta_y) + f'^2(\theta_y)}{f^2(\theta_x) + f'^2(\theta_x)}}\frac{H_1^{(1)}(kr(x, y))}{r(x, y)} \tag{3.39}$$

and

$$r(x, y) = [f^2(\theta_x) + f^2(\theta_y) - 2f(\theta_x)f(\theta_y)\cos(\theta_x - \theta_y)]^{1/2}. \tag{3.40}$$

The logarithmic singularities of S and K' are isolated if we write

$$M(x,y) = M_1(x,y)\log 4\left[\sin^2\frac{(\theta_x-\theta_y)}{2}\right] + M_2(x,y) \tag{3.41}$$

$$L'(x,y) = L'_1(x,y)\log 4\left[\sin^2\frac{(\theta_x-\theta_y)}{2}\right] + L'_2(x,y) \tag{3.42}$$

with

$$M_1(x,y) = -\frac{1}{2\pi}J_0(kr(x,y))\sqrt{f^2(\theta_y)+f'^2(\theta_y)} \tag{3.43}$$

$$M_2(x,y) = M(x,y) - M_1(x,y)\log 4\left[\sin^2\frac{(\theta_x-\theta_y)}{2}\right] \tag{3.44}$$

$$\begin{aligned} L'_1(x,y) = -\frac{k}{2\pi}[f^2(\theta_x) - f(\theta_x)f(\theta_y)\cos(\theta_y-\theta_x) + f'(\theta_x)f(\theta_y)\sin(\theta_y-\theta_x)] \\ \times\sqrt{\frac{f^2(\theta_y)+f'^2(\theta_y)}{f^2(\theta_x)+f'^2(\theta_x)}}\frac{J_1(kr(x,y))}{r(x,y)} \end{aligned} \tag{3.45}$$

$$L'_2(x,y) = L'(x,y) - L'_1(x,y)\log 4\left[\sin^2\frac{(\theta_x-\theta_y)}{2}\right]. \tag{3.46}$$

In particular, for $y = x$, using the expansion of H_0 [6], we have

$$M_2(x,x) = \left\{\frac{\mathrm{i}}{2} - \frac{\gamma}{\pi} - \frac{1}{2\pi}\log\left[\frac{k^2}{4}(f^2(\theta_x)+f'^2(\theta_x))\right]\right\}\sqrt{f^2(\theta_x)+f'^2(\theta_x)}$$

$$L'_2(x,x) = L'(x,x) = \frac{1}{2\pi}\frac{f^2(\theta_x)+2f'^2(\theta_x)-f(\theta_x)f''(\theta_x)}{f^2(\theta_x)+f'^2(\theta_x)}.$$

The Nyström method relies on approximating the integrals by quadrature formulae. To do this, we set the following expansion

$$M_1(x,y)\phi(y) \approx \sum_{k=0}^{2n-1} M_1(x,y_k)\phi_k^{(n)}(y_k)L_k^{(n)}(y) \tag{3.47}$$

where $\phi_k^{(n)}(y_k)$ are the values of ϕ at the $2n$ points y_k, $y_k := k(\pi/n)$, $k = 0,\ldots,2n-1$, and the $L_k^{(n)}(y)$ are the Lagrange basis for the trigonometric interpolation defined by

$$L_k^{(n)}(y) = \frac{1}{2\pi}\left\{1 + \sum_{l=0}^{n-1}\cos l(\theta_y - y_k) + \cos n(\theta_y - y_k)\right\}. \tag{3.48}$$

Hence we obtain

$$\int_0^{2\pi} \mathrm{d}\theta_y\, M_1(x,y)\log 4\left[\sin^2\frac{(\theta_x-\theta_y)}{2}\right]\phi(y) \approx \sum_{k=0}^{2n-1} M_1(x,y_k)\phi_k^{(n)}(y_k)R_k^{(n)}(x)$$

with $R_k^{(n)}(x) := \int_0^{2\pi} \mathrm{d}\theta_y \log 4[\sin^2((\theta_x - \theta_y)/2)]L_k^{(n)}(y)$.

This, after a few calculations, leads to [5]

$$R_k^{(n)}(x) = -\frac{2\pi}{n}\sum_{l=0}^{n-1}\frac{1}{l}\cos l(\theta_x - y_k) - \frac{\pi}{n^2}\cos n(\theta_x - y_k). \tag{3.49}$$

The regular parts of the integrals are calculated by a trapezoidal method. We use the same type of approximation in the second integral equation and we obtain the following approximate equations:

$$\phi^{(n)}(x) + \beta'(x)\sum_{k=0}^{2n-1}\left[R_k^{(n)}(x)M_1(x, y_k) + \frac{\pi}{n}M_2(x, y_k)\right]\phi_k^{(n)}(y_k) = -2\beta'(x)\mathrm{e}^{\mathrm{i}kf(\theta_x)\cos(\theta_x - \theta_k)} \tag{3.50}$$

$$\begin{aligned}\phi^{(n)}(x) &+ \tilde{\beta}'(x)\sum_{k=0}^{2n-1}\left[R_k^{(n)}(x)M_1(x, y_k) + \frac{\pi}{n}M_2(x, y_k)\right]\phi_k^{(n)}(y_k)\\ &+ \tilde{\beta}\sum_{k=0}^{2n-1}\left[R_k^{(n)}(x)L'_1(x, y_k) + \frac{\pi}{n}L'_2(x, y_k)\right]\phi_k^{(n)}(y_k)\\ &= -2\tilde{\beta}'\mathrm{e}^{\mathrm{i}kf(\theta_x)\cos(\theta_x - \theta_k)} + 2\mathrm{i}\tilde{\beta}(\nu_{x_j}\cdot k)\mathrm{e}^{\mathrm{i}kf(\theta_x)\cos(\theta_x - \theta_k)}.\end{aligned} \tag{3.51}$$

Finally we must solve finite linear equations obtained by writing equations (3.50) and (3.51) at quadrature points $\phi_j^n(x_j)$, $x_j := j(\pi/n)$, $j = 0, \ldots, 2n - 1$. These systems are:

$$\begin{aligned}\phi_j^{(n)}(x_j) &+ \beta'(x_j)\sum_{k=0}^{2n-1}\left[R_k^{(n)}(x)M_1(x_j, y_k) + \frac{\pi}{n}M_2(x_j, y_k)\right]\phi_k^{(n)}(y_k)\\ &= -2\beta'(x_j)\mathrm{e}^{\mathrm{i}kf(x_j)\cos(x_j - \theta_k)}\end{aligned} \tag{3.52}$$

$$\begin{aligned}\phi_j^{(n)}(x_j) &+ \sum_{k=0}^{2n-1}\tilde{\beta}'(x_j)\left[R_k^{(n)}(x)M_1(x_j, y_k) + \frac{\pi}{n}M_2(x_j, y_k)\right]\phi_k^{(n)}(y_k)\\ &+ \tilde{\beta}\sum_{k=0}^{2n-1}\left[R_k^{(n)}(x_j)L'_1(x_j, y_k) + \frac{\pi}{n}L'_2(x_j, y_k)\right]\phi_k^{(n)}(y_k)\\ &= -2\tilde{\beta}'(x_j)\mathrm{e}^{\mathrm{i}kf(x_j)\cos(x_j - \theta_k)} + 2\mathrm{i}\tilde{\beta}(\nu_{x_j}\cdot k)\mathrm{e}^{\mathrm{i}kf(x_j)\cos(x_j - \theta_k)}\end{aligned} \tag{3.53}$$

where $y_k := k(\pi/n)$, $k = 0, \ldots, 2n - 1$.

After solving these linear systems, we obtain the respective approximate solutions $\phi^{(n)}$ at any point $0 \leqslant x \leqslant 2\pi$, by replacing the values $\phi_k^{(n)}(y_k)$ at $y_k = k(\pi/n)$, $k = 0, \ldots, 2n-1$ in equations (3.50) and (3.53).

Finally, from this method we have the following result:

Theorem 5. Since equation (3.27) has a unique solution and the kernels $M_1(x, y)$ and $M_2(x, y)$ as well as the functions $\beta'(x)$ and $\mathrm{e}^{\mathrm{i}k\cdot x}$ are continuous, we have:

(i) the approximating linear system (3.52) has a unique solution for sufficiently large n;

(ii) for $n \to \infty$, the approximation solutions $\phi^{(n)}$ converge uniformly to the solution ϕ of the integral equation (3.27).

In addition, if $\beta'(x)$ and $\mathrm{e}^{\mathrm{i}k\cdot x}$ and S are analytic then the errors $\|\phi^{(n)} - \phi\|_\infty$ decrease exponentially; i.e. $\|\phi^{(n)} - \phi\|_\infty = O(\mathrm{e}^{-ns})$, where $s > 0$.

Of course, we have a similar convergence result for the second case.

4. Ambiguities with $\theta_{k'} = \theta_k + \delta$ at fixed energy

Now we present the numerical construction of equivalent scattering problems. We determine scatterers which give the same scattering amplitude for two fixed energies for all illumination angles. The observation angles are fixed in such a way that there is always a difference of a fixed value δ between the observation angle and the illumination angle. The scattering problems are treated with one discontinuity curve, without diffuse scattering (i.e. we consider the scattering amplitude A_0) and only in the Born approximation.

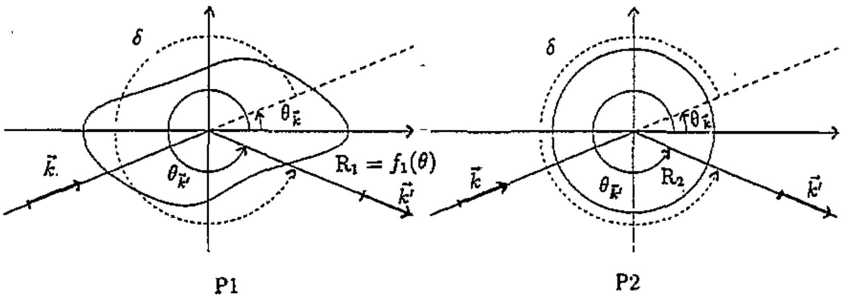

Figure 2. Ambiguities with $\theta_{k'} = \theta_k + \delta$ at fixed energy.

For the construction we assume that S_1 and S_2, the curves of two scattering problems, are star-shaped (see figure 2) with respect to the same centre. Their equations are $R_1 = f_1(\theta)$ and $R_2 = f_2(\theta)$ and their singular data are denoted $\beta_1(\theta)$, $\beta_1'(\theta)$, $\beta_2(\theta)$ and $\beta_2'(\theta)$. All these functions are 2π periodic. To summarize, P_1 is characterized by $\{R_1(\theta), \beta_1(\theta), \beta_1'(\theta)\}$ and P_2 by $\{R_2(\theta), \beta_2(\theta), \beta_2'(\theta)\}$. For the sake of simplicity, we choose a circle for the discontinuity curve of P_2.

Our goal is to construct the singular data $\beta_2(\theta)$ and $\beta_2'(\theta)$ supposing that $\{R_1(\theta), \beta_1(\theta), \beta_1'(\theta)\}$ and R_2 are given, such that we obtain the same amplitude for a given value of k for all the illumination angles, θ_k, and the observation angles, $\theta_{k'}$, satisfying the condition $\theta_{k'} = \theta_k + \delta$.

So, to obtain this, we start from the Born approximation relation (2.26) which gives for $\theta_{k'} = \theta_k + \delta$ the following expression:

$$\begin{aligned} A^{\mathrm{B}}_{0,k,\delta}(\theta_k) = 2 \int_0^{2\pi} \mathrm{d}\theta\, l(\theta) \mathrm{e}^{-2\mathrm{i}kf(\theta)\sin(\delta/2)\sin(\theta-\theta_k)} \\ + 4\mathrm{i}k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, g(\theta) \sin\left(\theta - \theta_k - \sin\frac{\delta}{2}\right) \mathrm{e}^{-2\mathrm{i}kf(\theta)\sin(\delta/2)\sin[\theta-\theta_k-\sin(\delta/2)]} \\ - 4\mathrm{i}k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, h(\theta) \cos\left(\theta - \theta_k - \sin\frac{\delta}{2}\right) \mathrm{e}^{-2\mathrm{i}kf(\theta)\sin(\delta/2)\sin[\theta-\theta_k-\sin(\delta/2)]} \end{aligned}$$

with

$$l(\theta) = \beta'(\theta)\sqrt{f^2(\theta) + f'^2(\theta)} \qquad g(\theta) = \beta(\theta) f(\theta) \qquad h(\theta) = \beta(\theta) f'(\theta). \tag{4.54}$$

One can notice that $A^{\mathrm{B}}_{0,k,\delta}(\theta_k)$ for fixed values of k and δ is a 2π periodic function of the illumination angle θ_k. Therefore we calculate the Fourier coefficients of $A^{\mathrm{B}}_{0,k,\delta}(\theta_k)$, A_m

and afterwards the coefficients $B_m^{(1),(2)}$ and $C_m^{(1),(2)}$ defined by the following expressions:

$$\begin{cases} B_m^{(1),(2)} = A_m^{(1),(2)} + A_{-m}^{(1),(2)} \\ C_m^{(1),(2)} = A_m^{(1),(2)} - A_{-m}^{(1),(2)} \end{cases} \qquad \text{with } m \in \mathbb{N}^{+*}.$$

The two problems P_1 and P_2 are equivalent if

$$\begin{aligned} A_0^{(1)} &= A_0^{(2)} \\ B_m^{(1)} &= B_m^{(2)} \qquad m \in \mathbb{N}^*. \\ C_m^{(1)} &= C_m^{(2)} \end{aligned}$$

After a few calculations we find the following infinite system:

$$\mu_0 \operatorname{Re}\beta_0^{(2)} + \nu_0 \operatorname{Re}\beta_0^{(2)} = A_0^{(1)} \tag{4.55}$$

$$\mu_m \operatorname{Re}\beta_m^{(2)} + \nu_m \operatorname{Re}\beta_m^{(2)} = B'^{(1)}_m \tag{4.56}$$

$$\mu_m \operatorname{Im}\beta_m^{(2)} + \nu_m \operatorname{Im}\beta_m^{(2)} = C'^{(1)}_m \tag{4.57}$$

with

$$\begin{cases} \mu_m = 2\pi R_2 J_m(2kR_2) \\ \nu_m = 2\pi k \sin\frac{\delta}{2} R_2 [J_{m+1}(2kR_2) - J_{m-1}(2kR_2)] \end{cases}$$

$$A_0^{(1)} = 2\int_0^{2\pi} \mathrm{d}\theta \left(l(\theta) J_0(2kf_1(\theta)) + 2k \sin\frac{\delta}{2} g(\theta) J_1(2kf_1(\theta)) \right)$$

$$\begin{aligned} B'^{(1)}_m = \int_0^{2\pi} & \mathrm{d}\theta\, l^{(1)}(\theta) J_m(2kf_1(\theta)) \cos m\theta \\ & + k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, g^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) - J_{m-1}(2kf_1(\theta))] \cos m\theta \\ & - k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, h^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) + J_{m-1}(2kf_1(\theta))] \sin m\theta \end{aligned}$$

$$\begin{aligned} C'^{(1)}_m = -\int_0^{2\pi} & \mathrm{d}\theta\, l^{(1)}(\theta) J_m(2kf_1(\theta)) \sin m\theta \\ & - k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, g^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) - J_{m-1}(2kf_1(\theta))] \sin m\theta \\ & - k \sin\frac{\delta}{2} \int_0^{2\pi} \mathrm{d}\theta\, h^{(1)}(\theta) [J_{m+1}(2kf_1(\theta)) + J_{m-1}(2kf_1(\theta))] \cos m\theta. \end{aligned}$$

We note that for m fixed there are two equations for four unknowns. To get four equations we choose two fixed energies. One could also choose one energy but two different radii and obtain three equivalent problems.

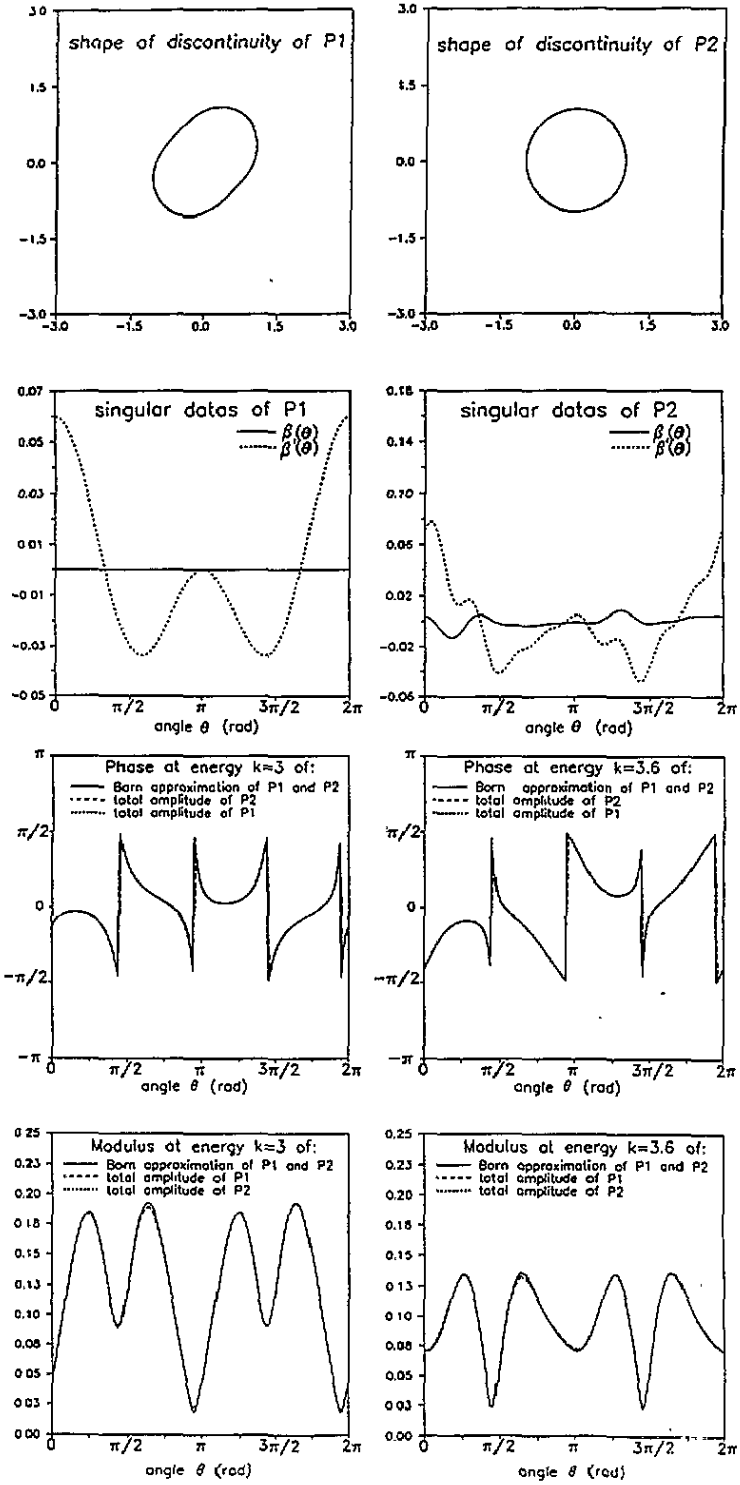

Figure 3. Scattering problems with $\delta = 1.605\pi$ with for P_1: $R_1 = 1 + 0.2\sin 2\theta$, $\beta_1 = 0$, $\beta_1' = 0.03(\cos\theta + \cos 2\theta)$ and P_2, $R_2 = 1$.

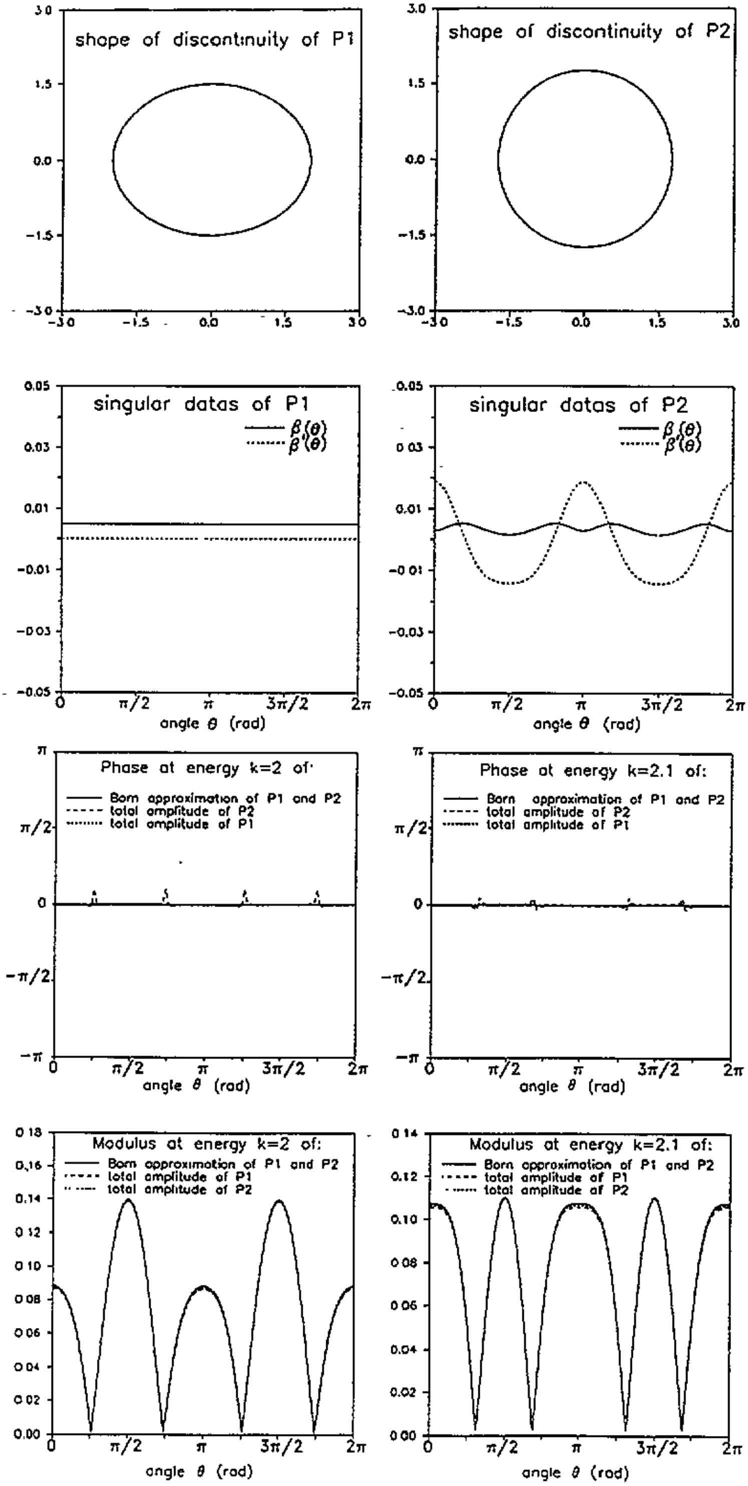

Figure 4. Scattering problems with $\delta = 1.605\pi$ with for P_1: $\beta_1 = 0.005$, $\beta_1' = 0$, $R_1 = 2.4/[1.44\cos^2(\theta) + 4\sin^2(\theta)]$ and for P_2: $R_2 = 1.6$.

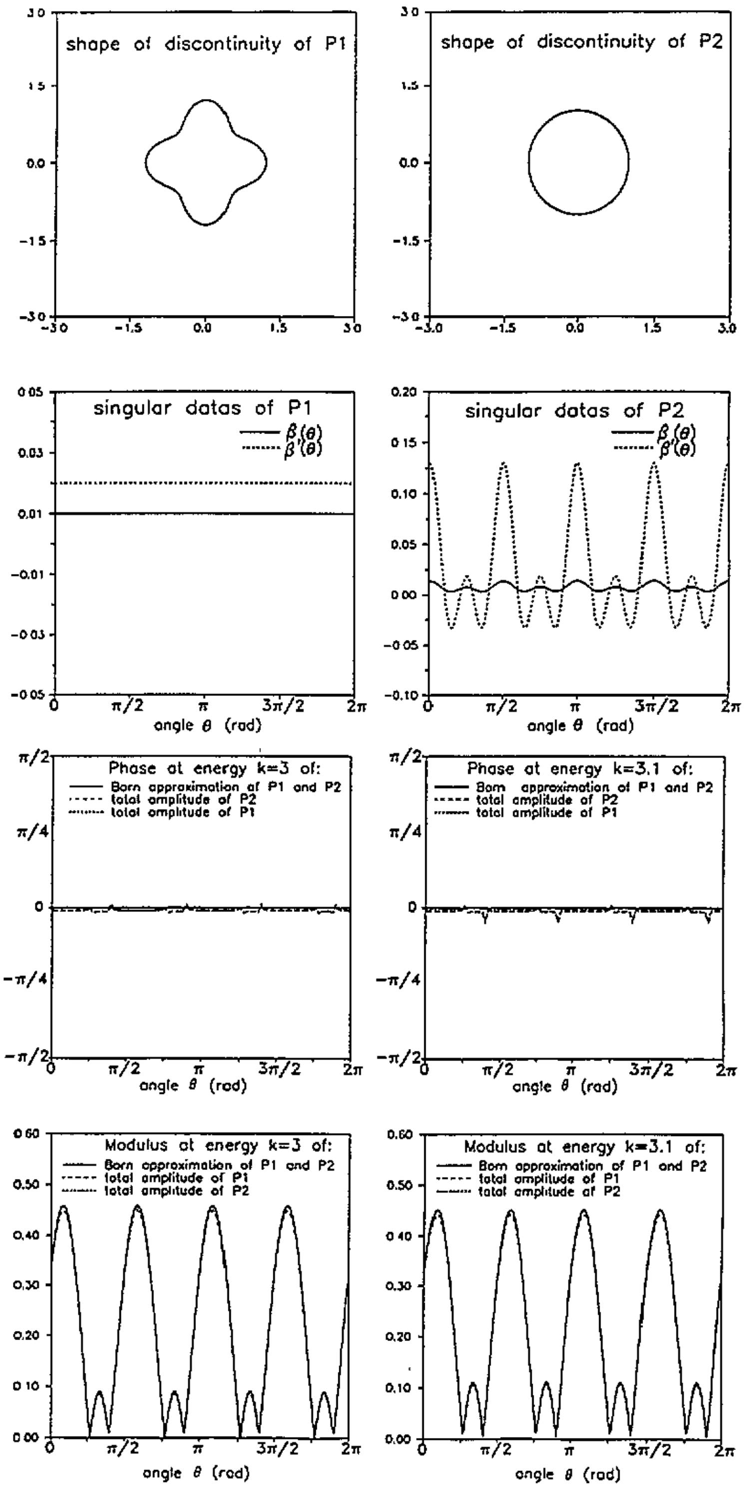

Figure 5. Scattering problems with $\delta = 4\pi/3$ with for P_1: $R_1 = 1 + 0.2\sin[4(\theta + \pi/8)]$, $\beta_1 = 0.01$, $\beta_1' = 0.02$ and for P_2, $R_2 = 1$.

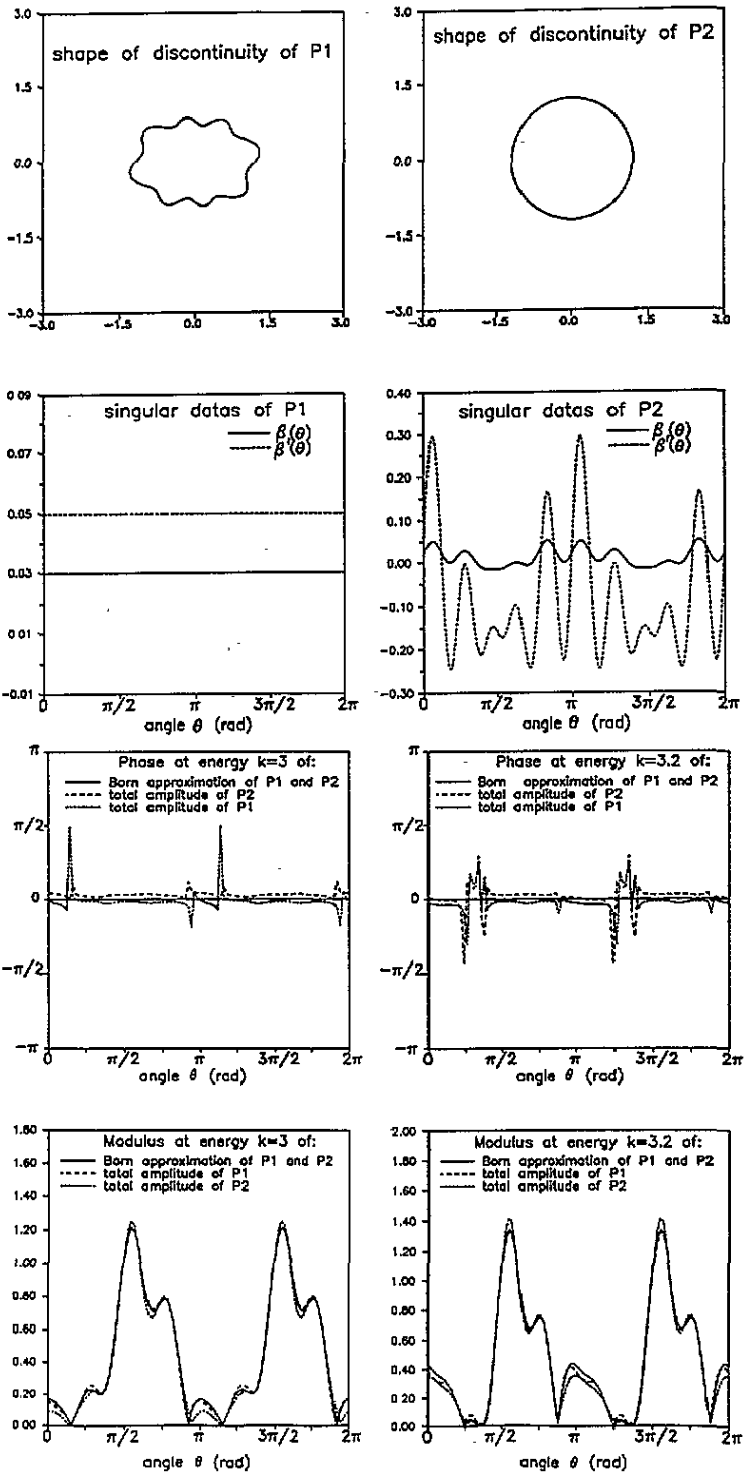

Figure 6. Scattering problems with $\delta = 1.6\pi/2$ with for P_1: $R_1 = (1+0.2\sin[4(\theta+\pi/4)])(1+0.1\sin[8(\theta+\pi/4)])$, $\beta_1 = 0.03$, $\beta_1' = 0.05$ and for P_2, $R_2 = 1.2$.

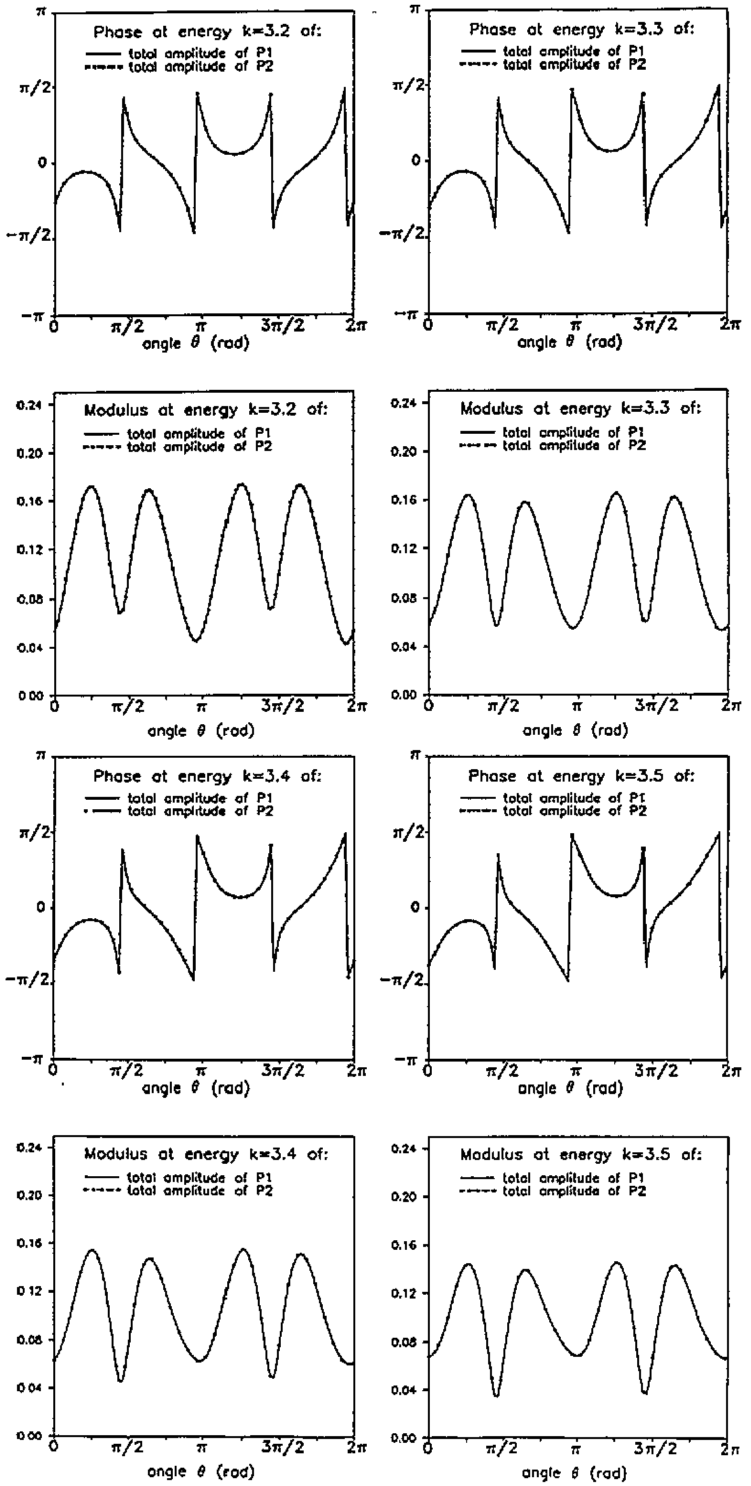

Figure 7. Example of variation of the energy between the values $k_1 = 3$ and $k_2 = 3.6$ for the equivalent problems defined in figure 3.

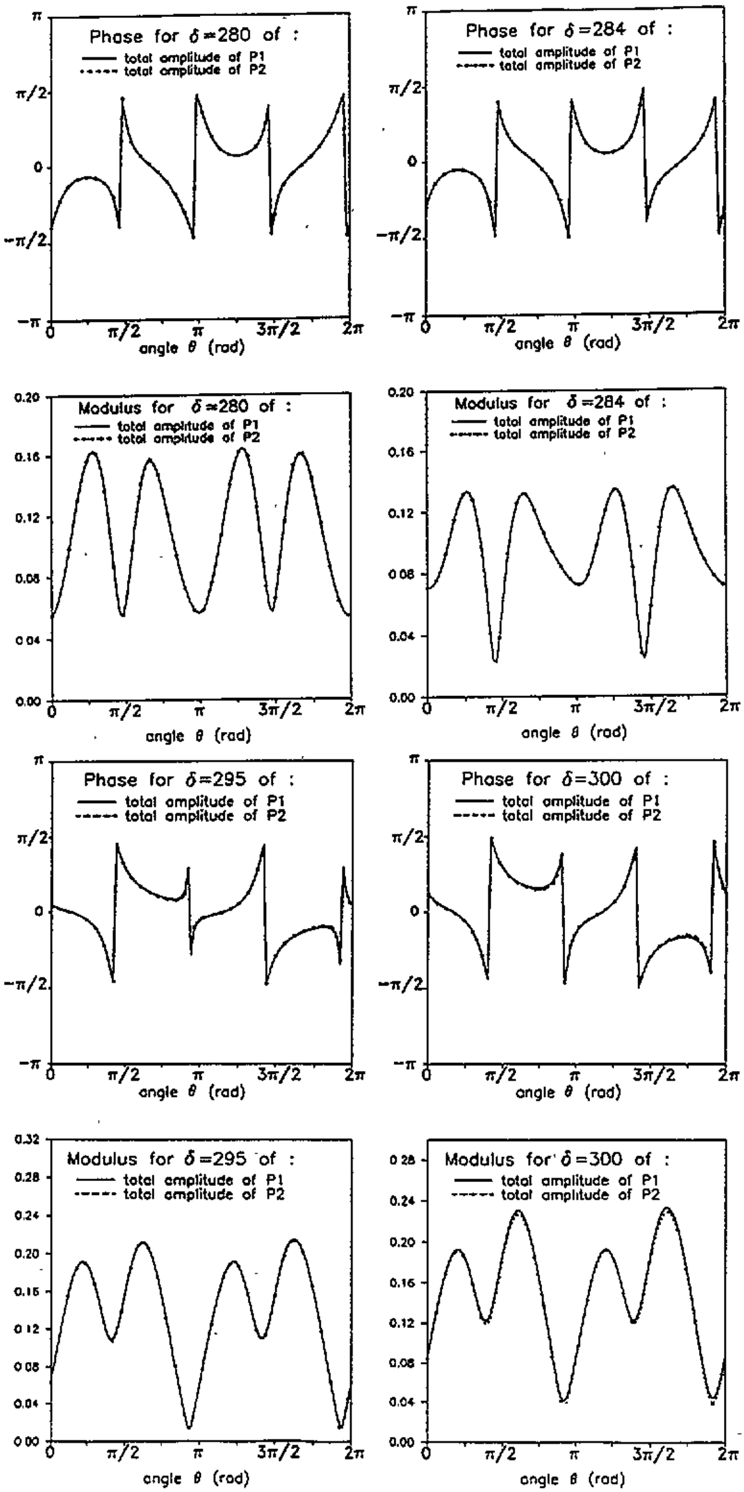

Figure 8. Example of variation of the δ difference between the observation angle and the illumination angle around the value $\delta_0 = 1.605\pi$ rad for the equivalent problems defined in figure 3.

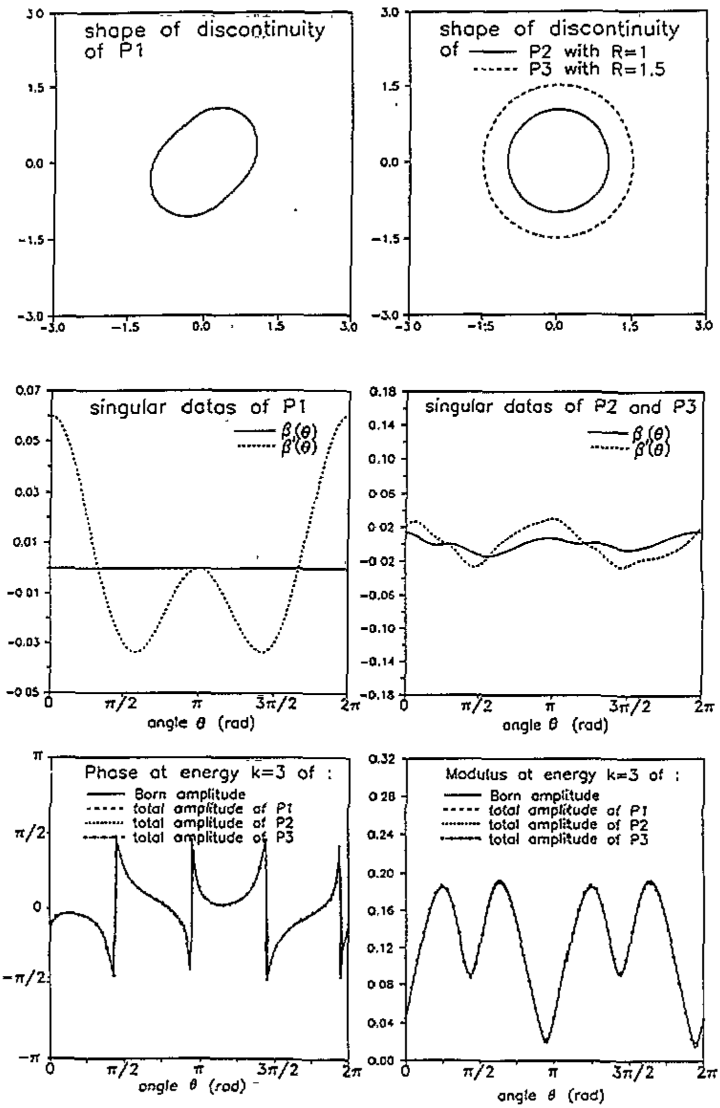

Figure 9. Scattering problems with $\delta = 1.605\pi$ with for P_1 : $R_1 = 1 + 0.2\sin 2\theta$, $\beta_1 = 0$, $\beta_1' = 0.03(\cos\theta + \cos 2\theta)$, for P_2, $R_2 = 1$ and for P_3, $R_3 = 1.5$.

The numerical examples are shown in the following figures. For each figure, we represent the characteristics of the original scattering situation P_1, i.e. $\{R_1(\theta),\ \beta_1(\theta),\ \beta_1'(\theta)\}$; the chosen circular discontinuous curve of P_2; and its singular data $\beta^{(2)}$ and $\beta'^{(2)}$ determined so that P_1 and P_2 are equivalent. We show the Born approximation of the scattering amplitude for the two chosen energies k_1 and k_2, with all illumination angles and observation angles such that a fixed difference of δ exists between the observation angle and the illumination angle. Further, we represent the total scattering amplitude of P_1 and P_2 for the same conditions (energies, illumination and observation angles) to verify that the equivalence constructed with the Born approximation is still true for the total amplitude. These exact scattering amplitudes are calculated by means of the Nyström method.

From these examples we can see that the exact scattering is still ambiguous although

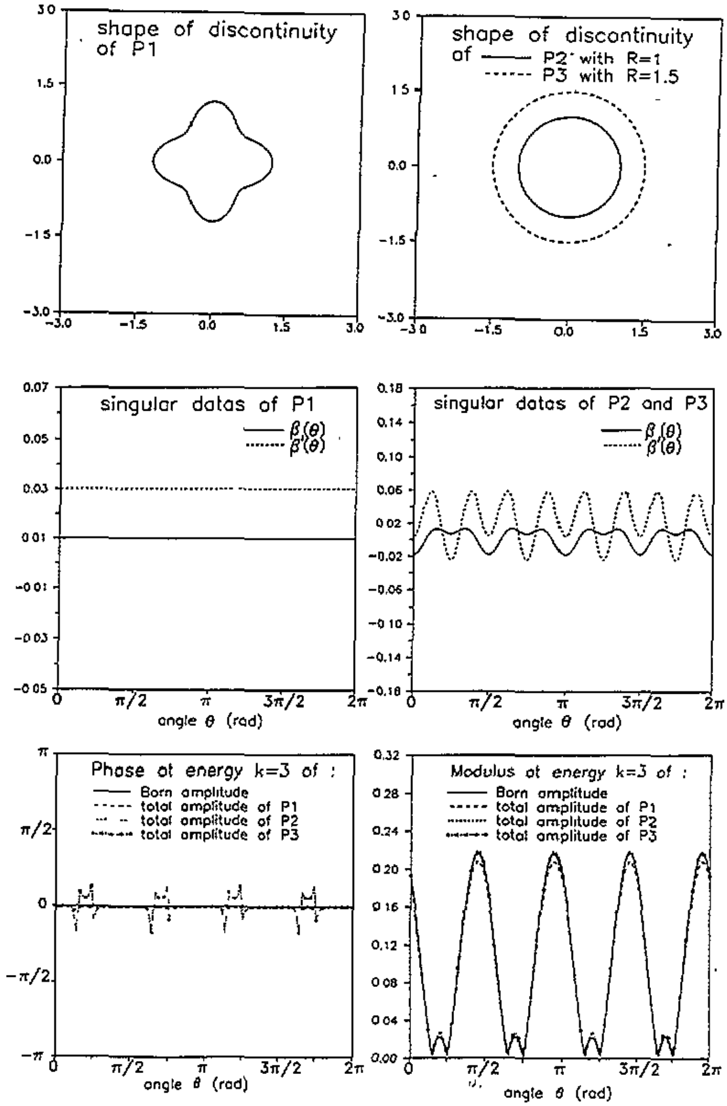

Figure 10. Scattering problems with $\delta = 1.605\pi$ with for P_1: $R_1 = 1 + 0.2\sin[4(\theta + \frac{1}{8}\pi)]$, $\beta_1 = 0.01$, $\beta_1' = 0.03$, for P_2, $R_2 = 1$ and for P_3, $R_3 = 1.5$.

the second chosen curve is fixed (a circle).

Figure 7 represents an example of the evolution of the amplitude according to the energy in the range of values between k_1 and k_2. The second evolution example (figure 8) is for a range of values of δ taken around δ_0. The values k_1, k_2 and δ_0 are the values for which the two equivalent problems have been determined. With the first example we can see that the ambiguity depends weakly on the energy. The second shows that the construction (for this example) depends weakly on the chosen difference between the observation angle and the illumination angle. A physical situation corresponding to this type of variation is realized when there is a fixed seat for the illumination and a fixed seat for the observation (but where the scatterer can turn around itself).

Figures 9 and 10 are examples where three equivalent problems are constructed for one given energy.

In conclusion, we also show another realistic measurement situation (see previous paper [3]) (fixed illumination angle, fixed observation angle with a scatterer turning around itself) where uniqueness is still not assured because we have weakened the hypothesis of Nachman's theorem by too much. These examples are another warning for the use of reconstruction methods in non-destructive sensing.

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